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Singular Loci of Varieties of Complexes, II

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In this paper we determine the singular loci of the irreducible components of the variety of complexes. © 2001 Academic Press

INTRODUCTION

Let $W_1, \dots, W_{h+1}$ be finite-dimensional vector spaces over some field k such that $\dim W_i = n_i$ for $1 \le i \le h+1$. Let Z be the affine space of all h-tuples of linear maps

$$(f_1, \dots, f_h): W_1 \xrightarrow{f_1} W_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{h-1}} W_h \xrightarrow{f_h} W_{h+1}.$$

The subvariety $\mathscr{C}$ of Z defined by

$$\mathscr{C} = \{(f_1, f_2, \dots, f_h) \in Z \mid f_{i+1} f_i = 0 \text{ for } 1 \le i \le h-1\}$$

is called the *variety of complexes*.

For positive integers $k_1, \dots, k_h$ such that $k_i + k_{i+1} \le n_{i+1}$ for $0 \le i \le h$ (where $k_0 = k_{h+1} = 0$), the varieties

$$V(k_1, \dots, k_h) = \{(f_1, \dots, f_h) \in \mathscr{C} \mid \operatorname{rank}(f_i) \le k_i \text{ for } 1 \le i \le h\}$$

are irreducible subvarieties of $\mathscr{C}$.

In this paper we determine (the irreducible components of) the singular locus of $V(k_1, \dots, k_h)$. Let $V = V(k_1, \dots, k_h)$,

$$V_i = V(k_1, \dots, k_{i-1}, k_i - 1, k_{i+1}, \dots, k_h) \qquad \text{for } 1 \le i \le h,$$

$$V_{j,j+1} = V(k_1, \dots, k_{j-1}, k_j - 1, k_{j+1} - 1, k_{j+2}, \dots, k_h) \qquad \text{for } 1 \le j \le h-1.$$

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We show the following

THEOREM 1. *The irreducible components of* $\operatorname{Sing} V$ *are* V_i, *with* $i \in \Omega$, *and* $V_{j,j+1}$, *with* $j \notin \Omega$, $j+1 \notin \Omega$, *where* Ω *is the set of all* $1 \le i \le h$ *such that* $k_{i-1} + k_i < n_i$ *and* $k_i + k_{i+1} < n_{i+1}$.

As a consequence, we obtain

THEOREM 2. *If* $V = V(k_1, \dots, k_h)$ *is an irreducible component of the variety of complexes* $\mathscr{C}$, *then the irreducible components of* $\operatorname{Sing} V$ *are* $V_{1,2}, \dots, V_{h-1,h}$.

For $h = 2$, these results are proved in [3].

The varieties $V(k_1, \dots, k_h)$ can be identified with opposite cells of certain Schubert varieties (see [4, 5]); thus, using Theorem 1, one can determine the irreducible components of these Schubert varieties.

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1. PRELIMINARIES

Let us fix the positive integers $h \ge 2$, $n_1, \dots, n_{h+1}$, and let $\mathbf{n} = (n_1, \dots, n_{h+1})$. Let $W_1, \dots, W_{h+1}$ be finite-dimensional vector spaces over a field k with $\dim W_1 = n_1, \dots, \dim W_{h+1} = n_{h+1}$. Let Z be the affine space of all h-tuples of linear maps

$$(f_1, \dots, f_h) \colon W_1 \xrightarrow{f_1} W_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{h-1}} W_h \xrightarrow{f_h} W_{h+1}.$$

The group $G_{\mathbf{n}} = GL(n_1) \times \cdots \times GL(n_{h+1})$ acts on Z by

$$(g_1, g_2, \dots, g_{h+1}) \cdot (f_1, f_2, \dots, f_h) = \left(g_2 f_1 g_1^{-1}, g_3 f_2 g_2^{-1}, \dots, g_{h+1} f_h g_h^{-1}\right).$$

The subvariety $\mathscr{C}$ of Z defined by

$$\mathscr{C} = \left\{(f_1, f_2, \dots, f_h) \in Z \mid f_{i+1} f_i = 0 \text{ for } 1 \le i \le h-1\right\}$$

is called the *variety of complexes*.

Let $\mathscr{K}_{\mathbf{n}}$ denote the set of all $\mathbf{k} = (k_1, \dots, k_h) \in \mathbb{Z}_+^h$, such that $k_i + k_{i+1} \le n_{i+1}$ for $0 \le i \le h$ (here $k_0 = k_{h+1} = 0$). For $\mathbf{k} = (k_1, \dots, k_h) \in \mathscr{K}_{\mathbf{n}}$, let

$$V(\mathbf{k}) = \left\{(f_1, \dots, f_h) \in \mathscr{C} \mid \operatorname{rank}(f_i) \le k_i \text{ for } 1 \le i \le h\right\}.$$

Let X_i be the $n_{i+1} \times n_i$ matrix of indeterminates, for $1 \le i \le h$, and let $k[X_1, \dots, X_h]$ be the polynomial ring over the indeterminates in $X_1 \cup \cdots \cup X_h$. For $\mathbf{k} \in \mathscr{K}_{\mathbf{n}}$, let $I(\mathbf{k})$ be the ideal in $k[X_1, \dots, X_h]$ generated by the entries of $X_{i+1} X_i$, for $1 \le i \le h-1$, and the $k_i + 1$ minors of X_i, for $1 \le i \le h$.

Let $G = GL(n)$. Let T be the maximal torus consisting of diagonal matrices in G, let B be the Borel subgroup consisting of the upper triangular matrices in G, and let $W = \mathcal{S}_n$ be the Weyl group of G. Let P_d, $1 \le d \le n - 1$, be the maximal parabolic subgroups in G, where

$$P_d = \left\{ A \in G \mid A = \begin{pmatrix} * & * \\ 0_{(n-d)\times d} & * \end{pmatrix} \right\}.$$

Consider the parabolic subgroup $Q = P_{a_1} \cap \cdots \cap P_{a_h}$ of G, i.e., the subgroup consisting of all elements of the form

$$\begin{pmatrix} A_1 & * & * & \cdots & * & * \\ 0 & A_2 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_h & * \\ 0 & 0 & 0 & \cdots & 0 & A_{h+1} \end{pmatrix},$$

where A_i is a matrix of size $(a_i - a_{i-1}) \times (a_i - a_{i-1})$ for $1 \le i \le h + 1$ (here $a_0 = 0$ and $a_{h+1} = n$). The Weyl group of Q is $W_Q = \mathcal{S}_{a_1} \times \mathcal{S}_{a_2 - a_1} \times \cdots \times \mathcal{S}_{n - a_h}$. The set of T-fixed points in G/Q for the action given by multiplication is precisely the set $\{e_{w,Q}, w \in W\}$, where for $w \in W$, $e_{w,Q}$ is the point in G/Q corresponding to the coset wQ. For $w \in W$, let $X_Q(w)$ be the *Schubert variety* in G/Q associated to wW_Q, i.e., the Zariski closure of $Be_{w,Q}$ in G/Q. It is well known that Schubert varieties are irreducible.

Let O^- be the set of elements of G of the form

$$\begin{pmatrix} I_1 & 0 & 0 & \cdots & 0 & 0 \\ * & I_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & * & * & \cdots & I_h & 0 \\ * & * & * & \cdots & * & I_{h+1} \end{pmatrix},$$

where I_j is the identity matrix of size $(a_j - a_{j-1}) \times (a_j - a_{j-1})$ for $1 \le j \le h + 1$ (here $a_0 = 0$ and $a_{h+1} = n$). Then the restriction of the canonical morphism $G \to G/Q$ to O^- is an open immersion, and the image of O^- is isomorphic to the *opposite big cell* in G/Q, i.e., with $B^- e_{\mathrm{id},Q}$, where B^- is the Borel subgroup opposite to B, consisting of the lower triangular matrices in G. Thus the set O^- is identified with the opposite big cell in G/Q. Any Schubert variety $X_Q(w) \subset G/Q$ has a non-empty intersection with the opposite big cell O^-, and $X_Q(w) \cap O^-$ is called the *opposite cell* in $X_Q(w)$.

The following result is proved in [5]:

THEOREM 1.1. *For* $\mathbf{k} \in \mathscr{K}_{\mathbf{n}}$, *the variety* $V(\mathbf{k})$ *is isomorphic to the opposite cell in a certain Schubert variety in* $GL(n)/Q$, *where* $n = n_1 + \cdots + n_{h+1}$ *and* $Q = P_{a_1} \cap \cdots \cap P_{a_h}$, $a_i = n_1 + \cdots + n_i$ *for* $1 \le i \le h$. *Furthermore*, $I(\mathbf{k})$ *defines the reduced scheme structure on* $V(\mathbf{k})$.

This isomorphism is the restriction of the map

$$\varphi\colon M(n_2 \times n_1) \times \cdots \times M(n_{h+1} \times n_h) \to O^-,$$

$$(A_1, \dots, A_h) \mapsto \begin{pmatrix} I_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ A_1 & I_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & A_2 & I_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_{h-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & A_{h-1} & I_h & 0 \\ 0 & 0 & 0 & \cdots & 0 & A_h & I_{h+1} \end{pmatrix} \pmod{Q}$$

to $V(\mathbf{k})$.

This result can also be obtained from the identification of *quiver varieties* (which are more general than the varieties $V(\mathbf{k})$ with $\mathbf{k} \in \mathscr{K}_{\mathbf{n}}$, discussed here) with opposite cells of Schubert varieties (see [4]).

It is easily seen that $V(\mathbf{k})$, with $\mathbf{k} = (k_1, \dots, k_h) \in \mathscr{K}_{\mathbf{n}}$, are all the closed irreducible $G_{\mathbf{n}}$-stable subvarieties of $\mathscr{C}$ and that $\mathscr{C} = \bigcup_{\mathbf{k} \in \mathscr{K}_{\mathbf{n}}} V(\mathbf{k})$.

Define the following partial order on the set $\mathscr{K}_{\mathbf{n}}$: for $\mathbf{k} = (k_1, k_2, \dots, k_h)$, $\mathbf{k}' = (k'_1, k'_2, \dots, k'_h)$ in $\mathscr{K}_{\mathbf{n}}, \mathbf{k} \ge \mathbf{k}' \Leftrightarrow k_i \ge k'_i$, $1 \le i \le h$. Clearly, $V(\mathbf{k}') \subseteq V(\mathbf{k})$ if and only if $\mathbf{k}' \le \mathbf{k}$. Hence we have the following

THEOREM 1.2. *The irreducible components of* $\mathscr{C}$ *are its subvarieties of the form* $V(\mathbf{k})$, *with* $\mathbf{k}$ *a maximal element of* $\mathscr{K}_{\mathbf{n}}$ (*with respect to the partial order above*).

The following result is also proved in [5]:

THEOREM 1.3. *Let* $\mathbf{k} \in \mathscr{K}_{\mathbf{n}}$.

1. $\dim V(\mathbf{k}) = \sum_{1 \le i \le h+1} (n_i - k_i)(k_{i-1} + k_i)$, *where* $k_0 = k_{h+1} = 0$.
2. $\operatorname{codim}_Z V(\mathbf{k}) = \sum_{i=1}^{h} (n_{i+1} - k_i)(n_i - k_i) + \sum_{i=1}^{h-1} k_i k_{i+1}$.

This theorem can also be deduced from [1].

Considering some fixed basis in each W_i, we have the identifications $W_i \cong k^{n_i}$ and $Z \cong M(n_2 \times n_1) \times \cdots \times M(n_{h+1} \times n_h)$, where $M(l \times m)$ denotes the affine space of matrices over k with l rows and m columns. Then $\mathscr{C}$ can be identified with the set of all points $(A_1, \dots, A_h)$ in

$M(n_2 \times n_1) \times \cdots \times M(n_{h+1} \times n_h)$ such that $A_2 A_1 = 0, \ldots, A_h A_{h-1} = 0$. For $\mathbf{k} \in \mathscr{K}_{\mathbf{n}}$, the variety $V(\mathbf{k})$ is identified with the set of all points $(A_1, \ldots, A_h)$ in $M(n_2 \times n_1) \times \cdots \times M(n_{h+1} \times n_h)$ such that $A_2 A_1 = 0, \ldots, A_h A_{h-1} = 0$, rank $A_1 \leq k_1, \ldots,$ rank $A_h \leq k_h$. Note that $V(\mathbf{k})$ is the closure of the $G_{\mathbf{n}}$-orbit in $M(n_2 \times n_1) \times \cdots \times M(n_{h+1} \times n_h)$ through a point $(A_1, \ldots, A_h)$ such that rank $A_1 = k_1, \ldots,$ rank $A_h = k_h$, and because of the action of $G_{\mathbf{n}}$, we may assume, without loss of generality, that

$$A_i = \begin{pmatrix} 0 & 0 \\ M_i & 0 \end{pmatrix}, \qquad \text{for } 1 \leq i \leq h,$$

where M_i is the $k_i \times k_i$ identity matrix.

2. SINGULAR LOCUS OF $V(k_1, \ldots, k_h)$

In this section we determine the singular points of $V(\mathbf{k})$, using the Jacobian criterion. Recall that, by Theorem 1.1, $I(\mathbf{k})$ is the defining ideal of $V(\mathbf{k})$.

Let $\mathbf{k} = (k_1, \ldots, k_h) \in \mathscr{K}_{\mathbf{n}}$ and $V = V(k_1, k_2, \ldots, k_h)$. For $1 \leq i \leq h$, let X_i denote the $n_{i+1} \times n_i$ matrix of variables, and let $\mathscr{M}_i$ be the set of all $k_i + 1$ minors of X_i. Then V is the set of zeros of the polynomials in $\mathscr{M}_1, \ldots, \mathscr{M}_h, X_2 X_1, \ldots, X_h X_{h-1}$. Let $x = (A_1, \ldots, A_h)$ be a point in V, A_i being an $n_{i+1} \times n_i$ matrix of rank at most k_i, and let J_x be the Jacobian matrix of V evaluated at x. The rows of J_x are indexed by the polynomials in the sets $\mathscr{M}_1, \ldots, \mathscr{M}_h, X_2 X_1, \ldots, X_h X_{h-1}$, while the columns of J are indexed by the entries in $X_1, \ldots, X_h$. The (M, α)th entry of J_x, where M is a minor in $\mathscr{M}_1 \cup \cdots \cup \mathscr{M}_h$ and α is an entry in $X_1 \cup \cdots \cup X_h$, is equal to $\pm$ the minor obtained from M by deleting the row and the column of α if α appears in M, and 0 otherwise. Let $f \in X_{i+1} X_i$ be the product of the rth row of X_{i+1} and the sth column of X_i, for some $1 \leq i \leq h-1$, $1 \leq r \leq n_{i+2}$, $1 \leq s \leq n_i$, and let α be an entry in $X_1 \cup \cdots \cup X_h$; then the (f, α)th entry of J_x is equal to the (j, s)th entry of X_i if α is the (r, j)th entry of X_i for some $1 \leq j \leq n_i$, the (r, l)th entry of X_{i+1} if α is the (l, s)th of X_i for some $1 \leq l \leq n_i$, and 0 otherwise.

The next lemma is obvious from the fact that the closure of the orbit through a point $x = (A_1, \ldots, A_h)$ in V such that rank $A_i = k_i$ for all $1 \leq i \leq h$ is V itself. We include the proof of this result to establish notations that will be used in the following, and for improved understanding of the Jacobian matrix of V.

LEMMA 2.1. *Let $x = (A_1, \ldots, A_h)$ be a point in V such that* rank *$A_i = k_i$ for all $1 \leq i \leq h$. Then x is a smooth point of V.*

Proof. Without loss of generality, we may assume that

$$A_i = \begin{pmatrix} 0 & 0 \\ M_i & 0 \end{pmatrix}, \qquad \text{for } 1 \le i \le h,$$

where M_i is the $k_i \times k_i$ identity matrix.

Let us denote J_x by just J.

Next we describe all the nonzero rows of J.

For $1 \le i \le h$, a row of J indexed by a minor $M \in \mathscr{M}_i$ of X_i is nonzero if and only if M contains the identity block M_i. The only nonzero element in such a row is equal to ± 1, placed in the column indexed by the upper right corner entry of M. When computing the rank of J, we may assume that such a ± 1 entry of J_x is actually equal to 1. Let $\mathscr{R}_i$ be the set of all minors in $\mathscr{M}_i$ containing M_i. Let Y_i be the set of all right upper corners of minors in $\mathscr{R}_i$, i.e., the set of all entries of X_i not contained in any of the rows or columns of M_i. Note that Y_i consists of the entries of the $(n_{i+1} - k_i) \times (n_i - k_i)$ right upper block of X_i, and hence its cardinality is $(n_{i+1} - k_i)(n_i - k_i)$. Also, $|\mathscr{R}_i| = (n_{i+1} - k_i)(n_i - k_i)$.

Now let f be an entry in $X_{i+1}X_i$ for some $1 \le i \le h-1$. In Figure 1, the blocks $X_i, X_{i+1}, X_{i+1}X_i$ are displayed as shown below:

X_i	
$X_{i+1}X_i$	X_{i+1}

Let f be the product of the rth row of X_{i+1} and the sth column of X_i, $1 \le r \le n_{i+2}$, $1 \le s \le n_i$. Then the row of J indexed by f is nonzero if and only if either the rth row of X_{i+1} is nonzero (i.e., it is one of the last k_{i+1} rows of X_{i+1}) or the sth column of X_i is nonzero (i.e., it is one of the first k_i columns of X_i). The set of all such f's in $X_{i+1}X_i$ can be written as $B_{i,i+1} \cup L_{i,i+1}$, where $B_{i,i+1}$ denotes the set of all entries in the last k_{i+1} rows of $X_{i+1}X_i$, and $L_{i,i+1}$ denotes the set of all entries in the first k_i columns of $X_{i+1}X_i$.

Now we find a maximal set of linearly independent rows of J among the set of rows described above.

The rows of J indexed by minors in $\mathscr{R}_1, \dots, \mathscr{R}_h$ are linearly independent, since their nonzero entries are in different columns (these columns are indexed by the entries in $Y_1, \dots, Y_h$, and each of them contains precisely one nonzero entry in the rows indexed by $\mathscr{R}_1, \dots, \mathscr{R}_h$).

For $1 \le i \le h-1$, let $K_{i,i+1} = B_{i,i+1} \cap L_{i,i+1}$ (note that $K_{i,i+1}$ is the $k_{i+1} \times k_i$ left lower block of $X_{i+1}X_i$). A row of J indexed by $f \in K_{i,i+1}$ contains precisely two nonzero entries (equal to 1), neither of which is in

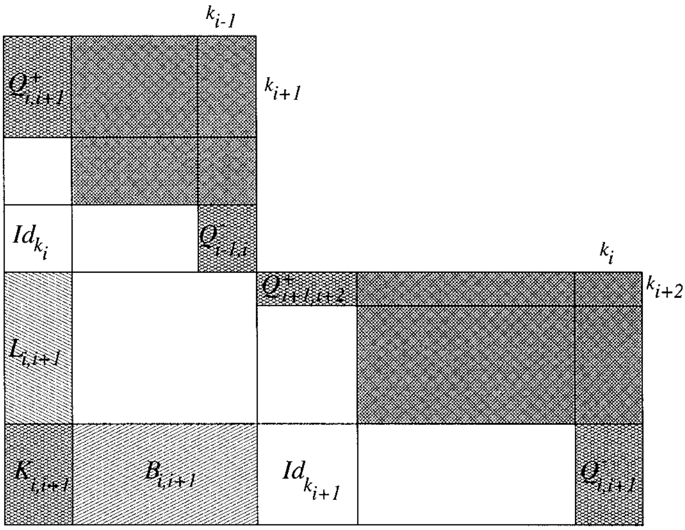

FIGURE 1

the same column with a nonzero entry of another nonzero row of J. The set of nonzero entries in the rows indexed by all $f \in K_{i,i+1}$ is in the columns indexed by the entries in the left upper block of X_i of size $k_{i+1} \times k_i$, which we denote by $Q^+_{i,i+1}$, and the right lower block of X_{i+1} of size $k_{i+1} \times k_i$, which we denote by $Q^-_{i,i+1}$.

A row of J indexed by $f \in (B_{i,i+1} \cup L_{i,i+1}) \setminus K_{i,i+1}$ contains precisely one nonzero entry (equal to 1) in a column indexed either by an element in the first k_{i+1} rows of Y_i or by an element in the last k_i columns of Y_{i+1}. Therefore these rows are identical to some of the rows indexed by minors in $\mathscr{R}_1, \dots, \mathscr{R}_h$.

Consequently, the rows of J indexed by minors in $\mathscr{R}_1, \dots, \mathscr{R}_h$ and entries of $K_{1,2}, \dots, K_{h-1,h}$ give a maximal set of linearly independent rows of J. Therefore,

$$\begin{aligned}\operatorname{rank} J &= |\mathscr{R}_1| + \cdots + |\mathscr{R}_h| + |K_{1,2}| + \cdots + |K_{h-1,h}| \\ &= (n_1 - k_1)(n_2 - k_2) + \cdots + (n_h - k_h)(n_{h+1} - k_h) \\ &\quad + k_1 k_2 + \cdots + k_{h-1} k_h \\ &= \operatorname{codim} V.\end{aligned}$$

This shows that x is a smooth point of V. ∎

Notation. For the rest of the paper,

$$J, Y_1, \dots, Y_h, Q^+_{1,2}, Q^-_{1,2}, \dots, Q^+_{h-1,h}, Q^-_{h-1,h}$$

will denote the Jacobian matrix, respectively the sets, associated, as in Lemma 2.1, to the point $(A_1, \ldots, A_h)$ in V with

$$A_i = \begin{pmatrix} 0 & 0 \\ M_i & 0 \end{pmatrix}, \qquad \text{for } 1 \leq i \leq h,$$

where M_i is the $k_i \times k_i$ identity matrix.

Remark 2.2. Note that the nonzero columns of J are the columns indexed by the entires in $Y_1, \ldots, Y_h$, $Q^+_{1,2}, Q^-_{1,2}, \ldots, Q^+_{h-1,h}, Q^-_{h-1,h}$. For $1 \leq j \leq h-1$, each column indexed by an entry in $Q^+_{j,j+1}$ is identical to a column indexed by the corresponding entry in $Q^-_{j,j+1}$. A maximal set of linearly independent columns of J consists of all distinct nonzero columns of J. We have $\operatorname{codim} V = |Y_1| + \cdots + |Y_h| + |Q^{\pm}_{1,2}| + \cdots + |Q^{\pm}_{h-1,h}|$ ($|Q^{\pm}_{j,j+1}|$ denotes the cardinality of each of $Q^+_{j,j+1}$ and $Q^-_{j,j+1}$). A nonzero column whose index is not included in the set of indices of the columns in a maximal set Σ of linearly independent columns of J is indexed by an entry in $Q^+_{j,j+1}$ or $Q^-_{j,j+1}$, for some j, and it is identical to a column whose index belongs to $Q^-_{j,j+1}$, respectively $Q^+_{j,j+1}$, and is among the indices of columns in Σ.

Remark 2.3. For $1 \leq j \leq h$, a maximal set of linearly independent columns of J indexed by entries in $X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_h$ consists of all such distinct nonzero columns. The cardinality of such a maximal set is $\operatorname{codim} V - |Y_j|$. The set of indices of the columns in such a maximal set must include $Q^+_{j-1,j}$ and $Q^-_{j,j+1}$.

Remark 2.4. For $1 \leq j \leq h-1$, a maximal set of linearly independent columns of J indexed by $X_1, \ldots, X_{j-1}, X_{j+2}, \ldots, X_h$ consists of all such distinct nonzero. The cardinality of such a maximal set is $\operatorname{codim} V - (|Y_j| + |Y_{j+1}| + |Q^{\pm}_{j,j+1}|)$. The set of indices of the columns in such a maximal set must include $Q^+_{j-1,j}$ and $Q^-_{j+1,j+2}$.

LEMMA 2.5. *Let $x = (A_1, \ldots, A_h)$ be a point of V such that* rank $A_j < k_j$ *for some j, $1 \leq j \leq h$. If x is smooth, then either $k_{j-1} + k_j = n_j$ or $k_j + k_{j+1} = n_{j+1}$ (here $k_0 = k_{h+1} = 0$).*

Proof. Let $t_i = \operatorname{rank} A_i$, for $1 \leq i \leq h$, with $t_j < k_j$. Without loss of generality, we may assume that

$$A_i = \begin{pmatrix} 0 & 0 \\ N_i & 0 \end{pmatrix}, \qquad \text{for } 1 \leq i \leq h,$$

where N_i is the $t_i \times t_i$ identity matrix. As noted in Remark 2.3, there are precisely $\operatorname{codim} V - |Y_j|$ linearly independent columns of J indexed by entries in $X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_h$. Since the nonzero columns of J_x are among the nonzero columns of J, the set T of linearly independent columns of J_x indexed by entries in $X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_h$ has cardinality at most $\operatorname{codim} V - |Y_j|$ (note that T consists of all distinct nonzero columns of J_x). Note that a column of J indexed by an entry in $X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_h$ is either identical to some column indexed by an entry in X_j or it is linearly independent of the columns indexed by entries in X_j. Since x is smooth, rank $J_x = \operatorname{codim} V$, so there exists a set S of cardinality at least $|Y_j|$ of linearly independent columns of J_x indexed by entries in X_j such that $S \cup T$ is a set of linearly independent columns of J_x. The only nonzero columns indexed by entries in X_j are actually indexed by entries in the first k_{j+1} rows of X_j and the last k_{j-1} columns of X_j (since rank $A_j < k_j$). The columns indexed by entries in $Q^-_{j-1,j} \cup Q^+_{j,j+1}$ cannot be in S, since they are in T (see Remark 2.3), and $S \cup T$ is a set of linearly independent columns of J_x. Thus S consists of columns indexed by entries in the first k_{j+1} rows of Y_j and the last k_{j-1} columns of Y_j. Since the cardinality of S must be at least $|Y_j|$, we deduce that the set of entries in the first k_{j+1} rows and the last k_{j-1} columns of Y_j must actually contain the set of entries in Y_j. This implies that either $k_{j-1} + k_j = n_j$ or $k_j + k_{j+1} = n_{j+1}$. ∎

LEMMA 2.6. *Let $x = (A_1, \ldots, A_h)$ be a point of V such that* rank $A_j < k_j$ *for some j, $1 \le j \le h$, and* rank $A_i = k_i$ *for $1 \le i \le h$, $i \ne j$. Then x is smooth if and only if either $k_{j-1} + k_j = n_j$ or $k_j + k_{j+1} = n_{j+1}$ (here $k_0 = k_{h+1} = 0$).*

Proof. In view of Lemma 2.5, we only have to show that if $k_{j-1} + k_j = n_j$ or $k_j + k_{j+1} = n_{j+1}$ then x is smooth. It is easily seen that the columns of J_x that are indexed by the elements in the disjoint sets

$$Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_h, Q^-_{1,2}, \ldots, Q^-_{j-2,j-1}, Q^+_{j+1,j+2}, \ldots, Q^+_{h-1,h}, \Gamma_j,$$

where Γ_j is the set of entries in the first k_{j+1} rows and the last k_{j-1} columns of X_j, are linearly independent. Note that Y_j is contained in Γ_j (since $k_{j-1} + k_j = n_j$ or $k_j + k_{j+1} = n_{j+1}$), and the cardinality of Γ_j is precisely $|Y_j| + |Q^-_{j-1,j}| + |Q^+_{j,j+1}|$. Since the number of these columns is $\operatorname{codim} V$, we conclude that rank $J_x = \operatorname{codim} V$, and hence x is smooth. ∎

Remark 2.7. An alternate proof of Lemma 2.6 can be obtained as follows. First note that if $k_{j-1} + k_j = n_j$, then $|L_{j-1,j}| = |\mathscr{R}_j|$, and if $k_j + k_{j+1} = n_{j+1}$, then $|B_{j,j+1}| = |\mathscr{R}_j|$. Let

$$\Delta_j = \begin{cases} L_{j-1,j} & \text{if } k_{j-1} + k_j = n_j, \\ B_{j,j+1} & \text{if } k_j + k_{j+1} = n_{j+1}, \\ L_{j-1,j} \text{ or } B_{j,j+1} & \text{if } k_{j-1} + k_j = n_j \text{ and } k_j + k_{j+1} = n_{j+1}. \end{cases}$$

We have $|\Delta_j| = |\mathscr{R}_j|$, and the rows of J indexed by elements in $\mathscr{R}_j$ are identical to the rows of J indexed by Δ_j. It is easily seen that the rows of J_x indexed by the codim V elements in the sets

$$\mathscr{R}_1, \ldots, \mathscr{R}_{j-1}, \Delta_j, \mathscr{R}_{j+1}, \ldots, \mathscr{R}_h, K_{1,2}, \ldots, K_{h-1,h}$$

are linearly independent, and hence x is smooth.

LEMMA 2.8. *Let $x = (A_1, \ldots, A_h)$ be a point of V such that if* rank $A_j < k_j$ *for some $1 \le j \le h$, then*

1. *either $1 \le j-1 \le h$ and* rank $A_{j-1} = k_{j-1}$, *or $1 \le j+1 \le h$ and* rank $A_{j+1} = k_{j+1}$;
2. *either $k_{j-1} + k_j = n_j$, or $k_j + k_{j+1} = n_{j+1}$ (here $k_0 = k_{h+1} = 0$).*

Then x is a smooth point of V.

Proof. Let Δ_j be as in Remark 2.7. Then the rows of J_x indexed by the codim V elements in the sets

$\mathscr{R}_i$, with i such that rank $A_i = k_i$,
Δ_j, with j such that rank $A_j < k_j$,
$K_{1,2}, \ldots, K_{h-1,h}$

are linearly independent. Therefore x is a smooth point of V. ∎

Remark 2.9. An alternative proof of Lemma 2.8 can be obtained as follows. For $1 \le j \le h$ with rank $A_j < k_j$, let Γ_j be as in the proof of Lemma 2.6. It is easily seen that the columns of J_x that are indexed by the elements in the disjoint sets

Y_i, with i such that rank $A_i = k_i$,
$Q^+_{i,i+1}$, with i such that rank $A_i = k_i$, rank $A_{i+1} = t_{i+1}$,
Γ_j, with j such that rank $A_j < k_j$

are linearly independent. Since the number of these columns is codim V, we conclude that rank $J_x =$ codim V, and hence x is smooth.

LEMMA 2.10. *Let $x = (A_1, \ldots, A_h)$ be a point of V such that* rank $A_j < k_j$ *and* rank $A_{j+1} < k_{j+1}$ *for some j, $1 \le j \le h-1$. Then x is a singular point of V.*

Proof. Let rank $A_i = t_i$, with $t_j < k_j, t_{j+1} < k_{j+1}$. Without loss of generality, we may assume that

$$A_i = \begin{pmatrix} 0 & 0 \\ N_i & 0 \end{pmatrix}, \qquad \text{for } 1 \le i \le h,$$

where N_i is the $t_i \times t_i$ identity matrix.

The nonzero columns of J_x are among the nonzero columns of J. Let T be the indices of a maximal set of linearly independent columns of J_x indexed by entries in $X_1, \dots, X_{j-1}, X_{j+2}, \dots, X_h$. Note that a column of J indexed by an entry in $X_1, \dots, X_{j-1}, X_{j+2}, \dots, X_h$ is either identical to some column indexed by an entry in $X_j \cup X_{j+1}$ or linearly independent of the columns indexed by entries in $X_j \cup X_{j+1}$. Thus there exists a set $S \subset X_j \cup X_{j+1}$ such that the columns of J_x indexed by elements of $S \cup T$ give a maximal set of linearly independent columns of J_x, and we have rank $J_x = |S| + |T|$.

As noted in Remark 2.4, there are precisely codim $V - (|Y_j| + |Y_{j+1}| + |Q^{\pm}_{j,j+1}|)$ linearly independent columns of J indexed by entries in $X_1, \dots, X_{j-1}, X_{j+2}, \dots, X_h$. Therefore $|T| \leq$ codim $V - (|Y_j| + |Y_{j+1}| + |Q^{\pm}_{j,j+1}|)$.

Now we show that $|S| < |Y_j| + |Y_{j+1}| + |Q^{\pm}_{j,j+1}|$. Each column indexed by an element of S contains precisely one nonzero entry in a row indexed by an entry of $X_j X_{j-1}, X_{j+1} X_j$, or $X_{j+2} X_{j+1}$ (since rank $A_j < k_j$ and rank $A_{j+1} < k_{j+1}$). The columns indexed by entries in $Q^-_{j-1,j}$ and $Q^+_{j+1,j+2}$ are identical to columns in T (see Remark 2.4). Therefore $S \cap Q^-_{j-1,j} = \varnothing$, $S \cap Q^+_{j+1,j+2} = \varnothing$, and $S \subset Y_j \cup Y_{j+1} \cup Q^+_{j,j+1} \cup Q^-_{j,j+1}$. The set of columns indexed by entries in $Q^+_{j,j+1}$ is the same as the set of columns indexed by entries in $Q^-_{j,j+1}$, so we may assume that $S \cap Q^-_{j,j+1} = \varnothing$. Thus we have $|S| = |S \cap Y_j| + |S \cap Y_{j+1}| + |S \cap Q^+_{j,j+1}|$. Obviously, $|S \cap Y_j| \leq |Y_j|$ and $|S \cap Y_{j+1}| \leq |Y_{j+1}|$ (note that one can have equalities here). On the other hand, we have $|S \cap Q^+_{j,j+1}| \leq t_j t_{j+1} < k_j k_{j+1} = |Q^{\pm}_{j,j+1}|$. Consequently, $|S| < |Y_j| + |Y_{j+1}| + |Q^{\pm}_{j,j+1}|$.

Thus rank $J_x <$ codim V, which shows that x is a singular point of V. ∎

Next we describe the singular locus of V.

For $1 \leq i \leq h$, let V_i denote the set of all points $(A_1, \dots, A_h)$ in V such that rank $A_i < k_i$, i.e.,

$$V_i = V(k_1, \dots, k_{i-1}, k_i - 1, k_{i+1}, \dots, k_h).$$

For $1 \leq j \leq h-1$, let $V_{j,j+1}$ denote the set of all points $(A_1, \dots, A_h)$ in V such that rank $A_j < k_j$ and rank $A_{j+1} < k_{j+1}$, i.e.,

$$V_{j,j+1} = V\big(k_1, \dots, k_{j-1}, k_j - 1, k_{j+1} - 1, k_{j+2}, \dots, k_h\big).$$

Note that $V_{j,j+1} = V_j \cap V_{j+1}$.

THEOREM 2.11. *The irreducible components of* Sing V *are the subvarieties of V of the form V_i, with $i \in \Omega$, and $V_{j,j+1}$, with $j \notin \Omega$, $j+1 \notin \Omega$, where Ω is the set of all $1 \leq i \leq h$ such that $k_{i-1} + k_i < n_i$ and $k_i + k_{i+1} < n_{i+1}$.*

Proof. Note that Sing V is a union of $G_{\mathbf{n}}$-orbit closures, i.e., subvarieties of V of the form $V(\mathbf{r})$, with $\mathbf{r} \in \mathscr{K}_{\mathbf{n}}$.

Next we determine for which $\mathbf{r} = (r_1, \ldots, r_h) \in \mathscr{K}_{\mathbf{n}}$ the subvariety $V(\mathbf{r})$ is singular. To check that $V(r_1, \ldots, r_h)$ is singular, it is enough to check that the point $x = (A_1, \ldots, A_h)$ with rank $A_i = r_i$ is singular.

If there is no i with $r_i < k_i$, by Lemma 2.1 $V(\mathbf{r})$ is not singular.

If there is precisely one index i with $r_i < k_i$, by Lemma 2.6 the subvariety $V(\mathbf{r}) = V_i$ is singular if and only if $i \in \Omega$.

Next, we analyze the subvarieties $V(\mathbf{r})$ with $r_i < k_i$ for more than one index i.

If there is some j with $r_j < k_j$ and $r_{j+1} < k_{j+1}$, then $V(\mathbf{r}) \subset V_{j,j+1}$. By Lemma 2.10 $V_{j,j+1}$ is singular, and therefore $V(\mathbf{r})$ is singular.

If for each j we have either $r_j = k_j$ or $r_{j+1} = k_{j+1}$, then by Lemma 2.8 $V(\mathbf{r})$ is singular only if $i \in \Omega$ for some i with $r_i < k_i$. But in this case $V(\mathbf{r}) \subset V_i$, and since V_i is singular, $V(\mathbf{r})$ is also singular.

Thus we have determined all the subvarieties of V of the form $V(\mathbf{r})$ with $\mathbf{r} \in \mathscr{K}_{\mathbf{n}}$, $\mathbf{r} \leq \mathbf{k}$, that are contained in Sing V. By taking the maximal such subvarieties (with respect to set inclusion), we obtain the required description for the irreducible components of Sing V. ∎

As a consequence, we obtain the following result that describes the singular loci of the irreducible components of the variety of complexes:

THEOREM 2.12. *If V is an irreducible component of the variety of complexes $\mathscr{C}$, then the irreducible components of* Sing V *are* $V_{1,2}, \ldots, V_{h-1,h}$.

Proof. In this case the set Ω is empty (by Theorem 1.2), and the result follows immediately from Theorem 2.11. ∎

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